ALGEBRA I

Semestral Examination

Instructions: All questions carry equal marks. Some questions are simpler than others. Therefore, read the entire question paper carefully before attempting to answer. Justify all your answers.

1. Let G be a finite group of order n and let m be a number that is relatively prime to n. Prove that the map $x \mapsto x^m$ on G is one-one. (Caution: This map may not be a group homomorphism!)

2. Let H be a normal subgroup of G, and let K be a subset of G which is closed under the group operation with $H \cap K = \{e\}$. Then prove that the set HK is a subgroup of G if and only if K is a subgroup of G.

3. Classify all groups of order 12 whose 3-Sylow subgroup is normal.

4. For a subgroup H of G, let $N(H)$ denote its normaliser. Prove that if H is a Sylow subgroup of G, then $N(N(H)) = N(H)$.

5. Let q be a prime number and let p be a prime factor of $q-1$. Prove that there exists a non-abelian subgroup of order pq in the symmetric group S_q.

6. Let N be a normal subgroup of G and let H be a Sylow subgroup of G. Prove or disprove: $H \cap N$ is a Sylow subgroup of N.